

WEIGHTED BADLY APPROXIMABLE VECTORS AND GAMES

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ABSTRACT. Let $d \geq 2$. We show that the set $\mathbf{Bad}(\mathbf{r})$ of \mathbf{r} -badly approximable vectors in \mathbb{R}^d is hyperplane absolute winning, hence is $1/2$ -winning for certain one-dimensional family of weights \mathbf{r} .

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1. INTRODUCTION

This paper is concerned with the study of weighted badly approximable vectors, which are natural generalization in high dimension of the classical badly approximable numbers. Its definition is as follows. For $d \in \mathbb{N}$, a tuple $\mathbf{r} = (r_1, \dots, r_d)$ belonging to the set

$$\mathcal{R}_d = \{\mathbf{r} = (r_1, \dots, r_d) : r_i \geq 0, \sum_{i=1}^d r_i = 1\},$$

and $\epsilon > 0$, set

$$\mathbf{Bad}_\epsilon(\mathbf{r}) = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq d} q^{r_i} \|qx_i\| \geq \epsilon\},$$

where $\|\cdot\|$ means the distance of a real number to its nearest integer. Put

$$\mathbf{Bad}(\mathbf{r}) = \bigcup_{\epsilon > 0} \mathbf{Bad}_\epsilon(\mathbf{r}),$$

which is called the set of \mathbf{r} -badly approximable vectors in \mathbb{R}^d . The tuple \mathbf{r} is called a *weight*. We also denote $\mathbf{Bad}(1/d, \dots, 1/d)$ simply by \mathbf{Bad}_d .

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Studying the intersection of $\mathbf{Bad}(\mathbf{r})$ for different weights \mathbf{r} is a very appealing subject which is undergoing rapid progress in recent years. In [5] Badziahin, Pollington and Velani proved a 30-year-old conjecture of Schmidt [17] that

$$\mathbf{Bad}\left(\frac{1}{3}, \frac{2}{3}\right) \cap \mathbf{Bad}\left(\frac{2}{3}, \frac{1}{3}\right) \neq \emptyset.$$

Actually the main result of [5] is much stronger. They proved that

Theorem 1.1 ([5]). *Let \mathcal{S} be a countable subset of \mathcal{R}_2 satisfying*

$$\text{dist}(\mathcal{S} \setminus \partial\mathcal{R}_2, \partial\mathcal{R}_2) > 0.$$

Then

$$\dim_H \bigcap_{\mathbf{r} \in \mathcal{S}} \mathbf{Bad}(\mathbf{r}) = 2,$$

where \dim_H means the Hausdorff dimension.

Afterwards, Beresnevich ([6]) proved that

Theorem 1.2 ([6]). *For any $d \geq 2$, let \mathcal{S} be a countable subset of \mathcal{R}_d satisfying*

$$(1.1) \quad \text{dist}(\mathcal{S} \setminus \partial\mathcal{R}_d, \partial\mathcal{R}_d) > 0.$$

Then

$$(1.2) \quad \dim_H \bigcap_{\mathbf{r} \in \mathcal{S}} \mathbf{Bad}(\mathbf{r}) = d.$$

In the 1960s, Schmidt introduced the (α, β) -game, which is played on metric spaces and whose winning sets, the α -winning sets ($\alpha \in (0, 1)$), have the following remarkable properties:

- the intersection of countably many α -winning sets is still α -winning,
- any α -winning subset of a Riemannian manifold is of full Hausdorff dimension.

Schmidt also proved that

Theorem 1.3. *For any $d \in \mathbb{N}$, \mathbf{Bad}_d is $1/2$ -winning.*

In [10], Kleinbock raised a question that whether $\mathbf{Bad}(\mathbf{r})$ is a winning set for any $d \geq 2$ and any weight $\mathbf{r} \in \mathcal{R}_d$. In view of the properties of the winning sets listed above, the conclusion (1.2) still holds without the technical condition (1.1) if Kleinbock's question has a positive answer. In [2], An answered Kleinbock's question positively in case $d = 2$ by proving the following theorem.

Theorem 1.4 ([2]). *For any $\mathbf{r} \in \mathcal{R}_2$, $\mathbf{Bad}(\mathbf{r})$ is $(24\sqrt{2})^{-1}$ -winning.*

When $d > 2$, nothing in this direction is known except Schmidt's classical result, namely Theorem 1.3.

In this paper, we use a variant of the (α, β) -game, namely the hyperplane absolute game introduced in [7] to study $\mathbf{Bad}(\mathbf{r})$ for certain weights \mathbf{r} in high dimension. Any hyperplane winning set is $1/2$ -winning (see Proposition 2.1). More details about the (α, β) -game, the hyperplane absolute game and their winning sets are given in Section 2. Recently, it was shown in [7] that for any $d \in \mathbb{N}$, \mathbf{Bad}_d is a hyperplane absolute winning set. It was also shown in [13] that for any $\mathbf{r} \in \mathcal{R}_2$, $\mathbf{Bad}(\mathbf{r})$ is a hyperplane absolute winning set.

For $d \geq 2$, consider a subset of \mathcal{R}_d defined as follows

$$(1.3) \quad \mathcal{R}'_d := \{\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{R}_d : \#\{i : r_i = \max_{1 \leq j \leq d} r_j\} \geq d - 1\}.$$

We prove the following theorem, which represents the first progress towards Kleinbock's question in high dimension.

Theorem 1.5. *For any $\mathbf{r} \in \mathcal{R}'_d$, $\mathbf{Bad}(\mathbf{r})$ is a hyperplane absolute winning set.*

An immediate corollary is as follows.

Corollary 1.6. *Let \mathcal{S} be a countable subset of \mathcal{R}'_d . Then*

$$\dim_H \bigcap_{\mathbf{r} \in \mathcal{S}} \mathbf{Bad}(\mathbf{r}) = d.$$

Relation with homogeneous dynamics. Based on works of Dani [8] and Kleinbock [10], it is now well known that weighted badly approximable vectors are closely related to bounded orbits in homogeneous dynamics. Precisely, set $G = \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$. For $\mathbf{r} \in \mathcal{R}_d$, let

$$F_{\mathbf{r}}^+ \text{ be the semigroup } \{\mathrm{diag}(e^{r_1 t}, \dots, e^{r_d t}, e^{-t}) : t > 0\};$$

and for $\mathbf{x} \in \mathbb{R}^d$, set

$$u_{\mathbf{x}} = \begin{pmatrix} I_d & \mathbf{x} \\ 0 & 1 \end{pmatrix} \in G.$$

Then we have the following Dani-Kleinbock correspondence [10, Theorem 2.5].

Proposition 1.7. *For $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ if and only if the orbit $F_{\mathbf{r}}^+ u_{\mathbf{x}} \Gamma$ is bounded in G/Γ .*

Put

$$E(F_{\mathbf{r}}^+) = \{x \in G/\Gamma : F_{\mathbf{r}}^+ x \text{ is bounded in } G/\Gamma\}.$$

As noted at the end of [4], using the methods developed in [4] one can gain information about the subset $E(F_{\mathbf{r}}^+) \subset G/\Gamma$ from properties of $\mathbf{Bad}(\mathbf{r}) \subset \mathbb{R}^d$. Hence, Theorem 1.5 may help to verify some special cases of [4, Conjecture 7.1].

Organization of the paper. In Section 2, we first recall the definitions and basic properties of various games. Then in Section 2.4, we reduce the proof of Theorem 1.5 to a concrete lemma (Lemma 2.4). In Section 3, we first attach a rational hyperplane \mathcal{H}_P to each rational point, then we define a decomposition of \mathbb{Q}^n using the attached hyperplane. This decomposition is a direct generalization of the decomposition introduced by An in [2], which plays a central role there. At the end of Section 3, we come up with the most important new ingredient in this paper, that is, we attach a line \mathcal{L}_P with well-chosen bounds on its defining coefficients to each rational point P in a certain class (Lemma 3.4). Only with such careful chosen bounds and lines, the proof goes through. The last two sections are devoted to proving Lemma 2.4.

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2. GAMES

In this section we first recall some basics of the (α, β) -game, the hyperplane absolute game, the hyperplane potential game and their winning sets. See [1], [4], [7], [9], [12], [14] for more details. Then we reduce the proof of Theorem 1.5 to a concrete lemma. We confine our discussion to subsets of a Euclidean space \mathbb{R}^d . Let $\rho(B)$ denote the radius of a closed ball B .

2.1. (α, β) -game. In [14], Schmidt introduced the (α, β) -game. Being played on \mathbb{R}^d , the game involves two parameters $\alpha, \beta \in (0, 1)$, a target set $S \subset \mathbb{R}^d$ and two players Alice and Bob. Let $i \geq 0$, at the i -th round, Bob chooses a closed ball B_i with $\rho(B_i) = \beta \rho(A_{i-1})$ (an arbitrary ball in case $i = 0$), and Alice chooses a closed ball $A_i \subseteq B_i$ with $\rho(A_i) = \alpha \rho(B_i)$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq A_0 \supseteq B_1 \supseteq A_1 \supseteq B_2 \supseteq \dots$$

S is called (α, β) -winning if Alice has a winning strategy ensuring that

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \emptyset,$$

regardless of how Bob chooses to play. For an $\alpha \in (0, 1)$, S is called α -winning if it is (α, β) -winning for any $\beta \in (0, 1)$.

No proper subset of \mathbb{R}^d is (α, β) -winning if $1 - 2\alpha + \alpha\beta \leq 0$ ([14, Lemma 5]). The α -winning sets enjoy many properties ([16]):

- (1) Given $\alpha, \alpha' \in (0, 1)$, if $\alpha \geq \alpha'$, then an α -winning set is also α' -winning. If $\alpha > 1/2$, then no proper subset of \mathbb{R}^d is α -winning.
- (2) The intersection of countably many α -winning sets is again an α -winning set.
- (3) If S is an α -winning set, then S is thick. Recall that a subset S of \mathbb{R}^d is *thick* if its intersection with any nonempty open subset of \mathbb{R}^d has full Hausdorff dimension.
- (4) Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bi-Lipschitz homeomorphism. If S is an α -winning set, then $\varphi(S)$ is α' -winning for some α' depending on φ and α .

As such, the (α, β) -game has been a powerful tool for proving full dimensionality (and non-emptiness) of fractal sets ([1], [2], [3], [4]).

2.2. Hyperplane absolute game. The hyperplane absolute game was introduced in [7]. It is played on a Euclidean space \mathbb{R}^d . Given a hyperplane \mathcal{H} and $\delta > 0$, denote by $\mathcal{H}^{(\delta)}$ the δ -neighborhood of \mathcal{H} ,

$$\mathcal{H}^{(\delta)} = \{\mathbf{x} \in \mathbb{R}^d : \text{dist}(\mathbf{x}, \mathcal{H}) \leq \delta\}.$$

For $\beta \in (0, 1/3)$, The β -hyperplane absolute game involves a parameter $\beta \in (0, 1/3)$, a target set $S \subset \mathbb{R}^d$ and two players Alice and Bob. Let $i \geq 0$, at the i -th round, Bob chooses a closed ball B_i of radius ρ_i such that $B_i \subseteq B_{i-1} \setminus \mathcal{H}_{i-1}^{(\delta_{i-1})}$ and $\rho_i \geq \beta\rho_{i-1}$ (an arbitrary ball in case $i = 0$), and Alice chooses a hyperplane neighborhood $\mathcal{H}_i^{(\delta_i)}$ with $\delta_i \leq \beta\rho_i$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$$

S is called β -hyperplane absolute winning (β -HAW for short) if Alice has a winning strategy ensuring that

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \emptyset,$$

regardless of how Bob chooses to play. S is called *hyperplane absolute winning* (HAW for short) if it is β -HAW for any $\beta \in (0, 1/3)$.

We have the following properties of β -HAW sets and HAW sets ([7], [11]),

- (1) Given $\beta, \beta' \in (0, 1/3)$, if $\beta \geq \beta'$, then any β' -HAW set is also β -HAW.
- (2) An HAW subset is α -winning for any α , $0 < \alpha < 1/2$.
- (3) The intersection of countably many HAW sets is again HAW.
- (4) Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a C^1 diffeomorphism. If S is an HAW set, then so is $\varphi(S)$.

We prove a strengthening of (2), which might be known to experts. It is of independent interest.

Proposition 2.1. *Given $\alpha, \beta \in (0, 1)$, if $1 - 2\alpha + \alpha\beta > 0$, then any HAW set is (α, β) -winning. In particular, any HAW set is $1/2$ -winning.*

Proof. For any $\alpha, \beta \in (0, 1)$ satisfying $1 - 2\alpha + \alpha\beta > 0$, write $\theta = 1 - 2\alpha + \alpha\beta$. Choose an $N \in \mathbb{N}$ satisfying

$$(2.1) \quad (\alpha\beta)^N < \frac{1}{3}\theta.$$

Write

$$(2.2) \quad \beta' = \frac{1}{2}(\alpha\beta)^N.$$

We are going to prove that β' -HAW implies (α, β) -winning, from which the conclusion follows. Let S be an HAW subset of \mathbb{R}^d . Let $B_i = B(y_i, \rho_i)$ (resp. $A_i = B(x_i, \alpha\rho_i)$) be Bob's (resp. Alice's) choice at the i -th round of the (α, β) -game. Then the sequences of balls $\{B(y_i, \rho_i)\}, \{B(x_i, \alpha\rho_i)\}$ should satisfy the following conditions,

$$(2.3) \quad \text{dist}(x_i, y_i) \leq (1 - \alpha)\rho_i, \quad \text{dist}(x_i, y_{i+1}) \leq \alpha(1 - \beta)\rho_i, \quad \rho_{i+1} = \alpha\beta\rho_i.$$

We are going to construct a corresponding β' -hyperplane absolute game, in which Bob's choice at the k -th round is the ball B_{kN} chosen by himself at the (kN) -th round in the (α, β) -game and Alice's choice at the k -th round is a hyperplane neighborhood $\mathcal{H}_k^{(\delta_k)}$ chosen according to her winning strategy. Once such a game is constructed, we obviously have the outcome point

$$x_\infty \in \bigcap_{k=0}^{\infty} B_{kN}$$

belonging to S . Thus S is (α, β) -winning. By definition, to construct such a game, we only need to make sure that:

- (a) $\rho_{(k+1)N} \geq \beta'\rho_{kN}$.
- (b) $B_{(k+1)N} \subset B_{kN} \setminus \mathcal{H}_k^{(\delta_k)}$.

Since $\rho_{(k+1)N} = (\alpha\beta)^N \rho_{kN}$, (a) follows directly from (2.2). Now we claim that if Alice chooses her ball A_i ($kN \leq i < (k+1)N$) as far away from \mathcal{H}_k as possible, then we have (b) as a consequence.

Indeed, for each $kN \leq i < (k+1)N$, Alice can choose x_i with

$$(2.4) \quad \text{dist}(x_i, \mathcal{H}_k) - \text{dist}(y_i, \mathcal{H}_k) = (1 - \alpha)\rho_i.$$

According to (2.3), no matter how Bob makes his choice, we always have

$$(2.5) \quad \text{dist}(y_{i+1}, \mathcal{H}_k) - \text{dist}(x_i, \mathcal{H}_k) \geq -\alpha(1 - \beta)\rho_i.$$

It follows from the inequalities (2.4) and (2.5) that

$$(2.6) \quad \text{dist}(y_{i+1}, \mathcal{H}_k) - \text{dist}(y_i, \mathcal{H}_k) \geq (1 - \alpha - \alpha(1 - \beta))\rho_i = \theta\rho_i.$$

Summing up the above inequalities (2.6) for all i ($kN \leq i < (k+1)N$), we get

$$(2.7) \quad \text{dist}(y_{(k+1)N}, \mathcal{H}_k) \geq \theta \sum_{i=kN}^{(k+1)N-1} \rho_i \geq \theta\rho_{kN}.$$

According to (2.1) and (2.2), we have

$$(2.8) \quad \rho_{(k+1)N} + \delta_k \leq ((\alpha\beta)^N + \beta')\rho_{kN} < \left(\frac{1}{3}\theta + \frac{1}{6}\theta\right)\rho_{kN} = \frac{1}{2}\theta\rho_{kN}.$$

Then, (b) follows from (2.7) and (2.8) immediately. \square

2.3. Hyperplane potential game. Being introduced in [9], the hyperplane potential game also defines a class of subsets of \mathbb{R}^d called *hyperplane potential winning* sets. By the following proposition these two classes of sets are indeed the same ([9, Theorem C.8]).

Proposition 2.2. *A subset S of \mathbb{R}^d is hyperplane potential winning if and only if it is hyperplane absolute winning.*

As such, the hyperplane potential game is a powerful tool for proving the HAW property since the hyperplane potential game is more flexible than the hyperplane absolute game in some aspects.

The hyperplane potential game involves two parameters $\beta \in (0, 1)$, $\gamma > 0$, a target set $S \subset \mathbb{R}^d$ and two players Alice and Bob. Let $i \geq 0$, at the i -th round, Bob chooses a closed ball B_i of radius ρ_i such that $\rho_i \geq \beta \rho_{i-1}$ (an arbitrary ball in case $i = 0$), and Alice chooses a countable family of hyperplane neighborhoods $\{\mathcal{H}_{i,k}^{(\delta_{i,k})} : k \in \mathbb{N}\}$ such that

$$(2.9) \quad \sum_{k=1}^{\infty} \delta_{i,k}^{\gamma} \leq (\beta \rho_i)^{\gamma}.$$

By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$$

S is called (β, γ) -hyperplane potential winning ((β, γ) -HPW for short) if Alice has a winning strategy ensuring that

$$\bigcap_{i=0}^{\infty} B_i \cap \left(S \cup \bigcup_{i=0}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{H}_{i,k}^{(\delta_{i,k})} \right) \neq \emptyset,$$

regardless of how Bob chooses to play. S is called *hyperplane potential winning* (HPW for short) if it is (β, γ) -HPW for any $\beta \in (0, 1)$ and $\gamma > 0$.

2.4. Reduction of Theorem 1.5. For the proof of Theorem 1.5, we may fix

$$d \geq 2, \text{ and } \mathbf{r} \in \mathcal{R}'_d$$

from now on. Then it is convenient to introduce the following notation.

Definition 2.3. *Let $B \subset \mathbb{R}^d$ be a closed ball, $\beta \in (0, 1)$ and $\gamma > 0$. Say a subset $S \subset \mathbb{R}^d$ is (B, β, γ) -HPW if Alice can win the (β, γ) -hyperplane potential game whenever Bob chooses B as his B_0 .*

Theorem 1.5 can be deduced from the following lemma.

Lemma 2.4. *For any closed ball $B_0 \subset \mathbb{R}^d$ of radius $\rho_0 \leq 1$, any $\beta \in (0, 1)$ and any $\gamma > 0$, the set $\mathbf{Bad}(\mathbf{r})$ is (B_0, β, γ) -HPW.*

Proof of Theorem 1.5 modulo Lemma 2.4. In view of Proposition 2.2, to prove Theorem 1.5, it suffices to prove that for any $\beta \in (0, 1)$ and any $\gamma > 0$, Alice has a winning strategy for the (β, γ) -hyperplane potential game played on \mathbb{R}^d with target set $\mathbf{Bad}(\mathbf{r})$. Denote the closed ball chosen by Bob at the i -th round as B_i with radius ρ_i . By [4, Remark 2.4], we may assume $\rho_0 \leq 1$ without loss of generality, which completes the proof of Theorem 1.5 modulo Lemma 2.4. \square

Remark 2.5. *The rest of the paper is devoted to proving Lemma 2.4. From now on, we fix a triple Φ which consists of*

- a closed ball $B_0 \subset \mathbb{R}^d$ of radius $\rho_0 \leq 1$,
- a number $\beta \in (0, 1)$,
- a number $\gamma > 0$.

3. ATTACHING A HYPERPLANE AND A LINE

Given a lattice L in \mathbb{R}^d , let $d(L) = \text{vol}(\mathbb{R}^d/L)$ denote the covolume of L . We shall need the following version of Minkowski's linear forms theorem ([16, Theorem 2C]).

Theorem 3.1. *Let $d \geq 2$. Given linearly independent linear forms l_1, \dots, l_d on \mathbb{R}^d , let (l_1, \dots, l_d) denote the linear transform from \mathbb{R}^d to itself generated by the linear forms l_1, \dots, l_d . For any lattice L in \mathbb{R}^d and positive numbers A_1, \dots, A_d satisfying*

$$A_1 \cdots A_d \geq |d(L)| \cdot |\det(l_1, \dots, l_d)|,$$

there exists $\mathbf{x} \in L \setminus \{\mathbf{0}\}$ such that

$$|l_1(\mathbf{x})| \leq A_1 \quad \text{and} \quad |l_i(\mathbf{x})| < A_i \quad (2 \leq i \leq d).$$

We make the following convention throughout this paper. Whenever we write a rational point $P \in \mathbb{Q}^d$ as

$$P = \frac{\mathbf{p}}{q} \quad \text{with } \mathbf{p} = (p_1, \dots, p_d),$$

we always mean

$$q > 0 \quad \text{and} \quad (p_1, \dots, p_d, q) = 1.$$

Such a form is unique. Then, the denominator q is a function of P . We write it as $q = q(P)$.

3.1. Attaching a hyperplane. For convenience, we set

$$(3.1) \quad s = s(\mathbf{r}) = \max_{1 \leq i \leq d} r_i, \quad \text{and choose } i_0 \in \{j : r_j = \min_{1 \leq i \leq d} r_i\}.$$

For each rational point $P = \mathbf{p}/q \in \mathbb{Q}^d$, we define a lattice

$$(3.2) \quad \Lambda_P = \{b \frac{\mathbf{p}}{q} + \mathbf{z} : b \in \mathbb{Z}, \mathbf{z} \in \mathbb{Z}^d\}.$$

Since $[\Lambda_P : \mathbb{Z}^d] = q$, we have $d(\Lambda_P) = 1/q$. Hence $d(\Lambda_P^*) = q$, where Λ_P^* means the dual lattice of Λ_P . Note that

$$\Lambda_P^* = \{\mathbf{a} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{p} \in q\mathbb{Z}\}.$$

Choose and fix

$$(3.3) \quad \mathbf{a}_P \in X_P = [-q^{r_1}, q^{r_1}] \times \cdots \times [-q^{r_d}, q^{r_d}] \cap \Lambda_P^* \setminus \{\mathbf{0}\}.$$

The non-emptiness of X_P is ensured by Theorem 3.1.

Define an affine form

$$(3.4) \quad F_P(\mathbf{x}) = \mathbf{a}_P \cdot \mathbf{x} + C_P, \quad \text{where } C_P = -q^{-1} \mathbf{a}_P \cdot \mathbf{p} \in \mathbb{Z},$$

and a hyperplane

$$(3.5) \quad \mathcal{H}_P = \{\mathbf{x} \in \mathbb{R}^d : F_P(\mathbf{x}) = 0\}.$$

It is obvious that $P \in \mathcal{H}_P$.

Write $\mathbf{a}_P = (a_1, \dots, a_d)$. We define two \mathbb{N} -valued functions on \mathbb{Q}^d ,

$$(3.6) \quad \xi_P = \xi(P) = \max\{|a_i| : 1 \leq i \leq d\}$$

and

$$(3.7) \quad H(P) = q(P)\xi(P).$$

According to (3.3),

$$(3.8) \quad \xi_P \leq q(P)^s \quad \text{and} \quad H(P) \leq q(P)^{1+s}.$$

Note that

$$(3.9) \quad |a_i| \leq \min\{q(P)^{r_i}, \xi_P\} \quad \text{for any } 1 \leq i \leq d$$

3.2. Constants and subdivisions. In this subsection, we introduce some constants and subdivisions. For $n \geq 1$, let \mathcal{B}_n be the set of closed balls defined by

$$(3.10) \quad \mathcal{B}_n = \{B \subset B_0 : \beta R^{-n} \rho_0 < \rho(B) \leq R^{-n} \rho_0\},$$

where R is a positive number satisfying

$$(3.11) \quad (R^\gamma - 1)^{-1} \leq \left(\frac{\beta^2}{2}\right)^\gamma.$$

Note that this implies

$$(3.12) \quad R > 2\beta^{-2} > \max\{2, \beta^{-1}\},$$

which shows that those \mathcal{B}_n are mutually disjoint.

Now we define a decomposition of \mathbb{Q}^d . Write

$$(3.13) \quad c = \frac{1}{8}d^{-2}\rho_0 R^{-18d^2} \quad \text{and} \quad H_n = dc\rho_0^{-1}R^n.$$

Put

$$(3.14) \quad \mathcal{P}_n = \{P \in \mathbb{Q}^d : H_n \leq H(P) < H_{n+1}\}.$$

Set

$$(3.15) \quad \mathcal{P}_{n,1} = \{P \in \mathcal{P}_n : H_n^{\frac{1}{1+s}} \leq q(P) < H_n^{\frac{1}{1+s}} R^{12d^2}\}.$$

For $k \geq 2$, write

$$(3.16) \quad Q_{n,k} = H_n^{\frac{1}{1+s}} R^{d(k-2)+12d^2}$$

and set

$$(3.17) \quad \mathcal{P}_{n,k} = \{P \in \mathcal{P}_n : Q_{n,k} \leq q(P) < Q_{n,k+1}\}.$$

By definition, those $\mathcal{P}_{n,k}$ are mutually disjoint. The following lemma summarizes some basic properties of this decomposition.

Lemma 3.2.

- (1) $\mathbb{Q}^d = \bigsqcup_{n=1}^{\infty} \bigsqcup_{k=1}^{n-1} \mathcal{P}_{n,k}$.
- (2) For $P \in \mathcal{P}_{n,k}$ with $k \geq 2$, we have

$$(3.18) \quad \psi_P := q^{-1-s}H(P) \leq R^{-dk-10d^2}.$$

Proof. As $H_1 < 1$, it follows that $\mathbb{Q}^d = \bigsqcup_{n=1}^{\infty} \mathcal{P}_n$. By definition, $\mathcal{P}_n = \bigsqcup_{k=1}^{\infty} \mathcal{P}_{n,k}$. Hence to prove (1), it suffices to prove that $\mathcal{P}_{n,k} = \emptyset$ for $k \geq n$. We argue by contradiction. Note that $H_1 < 1$, so we may assume that $\mathcal{P}_{n,k} \neq \emptyset$ for some n and k with $k \geq n \geq 1$. Taking $P \in \mathcal{P}_{n,k}$, then by definition we have

$$H_n^{\frac{1}{1+s}} R^{d(k-2)+12d^2} = Q_{n,k} \leq q(P) < H_{n+1} = H_n R.$$

Since $H_n \leq R^n$ and $s \leq 1/(d-1)$, we have

$$R^n \geq H_n \geq R^{\frac{s+1}{s}(d(k-2)+12d^2-1)} \geq R^{d(k-2)+12d^2-1} \geq R^{d^2k}.$$

Then $k < n$, which leads to a contradiction.

The inequality (3.18) is verified by a direct computation,

$$\psi_P = q^{-1-s}H(P) \leq Q_{n,k}^{-1-s}H_{n+1} = R^{1-(1+s)(d(k-2)+12d^2)} \leq R^{-dk-10d^2}.$$

□

Remark 3.3. We use k_P to denote the unique number such that $P \in \mathcal{P}_{*,k_P}$, which is well-defined by Lemma 3.2(1).

3.3. Attaching a line. In this subsection, we attach a suitable rational line to each rational point P with $k_P \geq 2$. We begin with a lemma.

Lemma 3.4. *Let $P = \mathbf{p}/q \in \mathbb{Q}^d$ and $\mathbf{a}_P = (a_1, \dots, a_d)$. Set*

$$(3.19) \quad \mathbf{w}_P = (w_1, \dots, w_d) \text{ with } w_i = 1 \text{ for } i \neq i_0, \text{ and } w_{i_0} = \psi_P,$$

where i_0 is given in (3.1). Then there exists $\mathbf{v} = (v_1, \dots, v_d) \in \Lambda_P \setminus \{\mathbf{0}\}$ such that

$$(3.20) \quad |v_i| \leq (d-1)w_i q^{-r_i} \quad \text{for each } 1 \leq i \leq d.$$

Proof. Let $j \in [1, d]$ be such that

$$w_j |a_j| q^{-r_j} = \max\{w_i |a_i| q^{-r_i} : 1 \leq i \leq d\}.$$

Set

$$\Pi_j := \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : |x_i| \leq w_i q^{-r_i} \text{ for } i \neq j, |\mathbf{a}_P \cdot \mathbf{v}| < 1\}.$$

We claim that

$$(3.21) \quad \text{Vol}(\Pi_j) \geq q^{-1}.$$

There are two cases.

- If $j \neq i_0$, then $\text{Vol}(\Pi_j) = |a_j|^{-1} \prod_{i \neq j} w_i q^{-r_i} \geq q^{-1}$.
- If $j = i_0$, then $\text{Vol}(\Pi_j) = |a_j|^{-1} \prod_{i \neq j} w_i q^{-r_i} \geq \xi_P^{-1} \psi_P q^{-1+s} \geq q^{-1}$.

In view of Theorem 3.1, (3.21) and the fact $d(\Lambda_P) = 1/q$, there exists

$$\mathbf{v} = (v_1, \dots, v_d) \in \Pi_j \cap \Lambda_P \setminus \{\mathbf{0}\}.$$

Since $\mathbf{a}_P \in \Lambda_P^*$, $|\mathbf{a}_P \cdot \mathbf{v}| < 1$ implies $|\mathbf{a}_P \cdot \mathbf{v}| = 0$. Hence we have the following estimate,

$$|v_j| \leq |a_j|^{-1} \sum_{i \neq j} |a_i v_i| \leq |a_j|^{-1} \sum_{i \neq j} w_i \frac{|a_i|}{q^{r_i}} \leq (d-1)w_j q^{-r_j},$$

which completes the proof. \square

According to Lemma 3.4, for each rational point P with $k_P \geq 2$, the set of vectors \mathbf{v} satisfying (3.20) is nonempty. We choose and fix one as \mathbf{v}_P . Then we define a rational line passing through P by

$$(3.22) \quad \mathcal{L}_P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} - \frac{\mathbf{p}}{q} = \lambda \mathbf{v}_P, \lambda \in \mathbb{R}\}.$$

4. A KEY PROPOSITION

It is easily checked that

$$\mathbf{Bad}_\epsilon(\mathbf{r}) = \mathbb{R}^d \setminus \bigcup_{P \in \mathbb{Q}^d} \Delta_\epsilon(P),$$

where for $P = (p_1/q, \dots, p_d/q) \in \mathbb{Q}^d$,

$$(4.1) \quad \Delta_\epsilon(P) = \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : |x_i - \frac{p_i}{q}| < \frac{\epsilon}{q^{1+r_i}}, i = 1, 2, \dots, d\}.$$

Define a partial order " $<$ " on the set \mathbb{Q}^d by

$$P < P' \iff \Delta_c(P) \subsetneq \Delta_c(P'),$$

where the constant c is given in (3.13).

Definition 4.1. *A rational point P is called maximal if P is a maximal element for the partial order $<$. Let \mathcal{C} denote the set of maximal points.*

Remark 4.2. *Note that the partial order defined as above depends on the constant c , which is determined by the triple Φ . As Φ has already been fixed, we omit the dependence in the definition.*

The following lemma is easy but important.

Lemma 4.3.

$$\bigcup_{P \in \mathbb{Q}^d} \Delta_c(P) = \bigcup_{P \in \mathcal{C}} \Delta_c(P).$$

Proof. We need to show that: for any P , there is a maximal element Q such that $P < Q$. Indeed it follows directly from the definition that the set

$$(4.2) \quad \{P' \in \mathbb{Q}^d : P < P'\}$$

is finite. Assume the contrary that there is no maximal element Q such that $P < Q$, then it follows that the set (4.2) is infinite, which leads to a contradiction. \square

For a closed ball $B \in \mathcal{B}_n$, set

$$\mathcal{C}_{n+k,k}(B) = \{P \in \mathcal{P}_{n+k,k} \cap \mathcal{C} : \Delta_c(P) \cap B \neq \emptyset\}.$$

The following is a key proposition needed in the proof of Lemma 2.4.

Proposition 4.4. *Let $n \geq 1$, $B \in \mathcal{B}_n$ and $k \geq 1$. Then*

$$\mathcal{C}_{n+k,k}(B) \subset \bigcap_{P \in \mathcal{C}_{n+k,k}(B)} \mathcal{H}_P.$$

To prove Proposition 4.4, it suffices to show that $P_1 \in \mathcal{H}_{P_2}$ for any $P_1, P_2 \in \mathcal{C}_{n+k,k}(B)$. We pick

$$P_1 = \frac{\mathbf{p}_1}{q_1}, P_2 = \frac{\mathbf{p}_2}{q_2} \in \mathcal{C}_{n+k,k}(B), \text{ with } \mathbf{p}_j = (p_{1,j}, \dots, p_{d,j}) \text{ where } j = 1, 2,$$

and write

$$\mathbf{a}_{P_j} = (a_{1,j}, \dots, a_{d,j}) \text{ where } j = 1, 2.$$

Before proving Proposition 4.4, we give the following useful estimate.

Lemma 4.5. *For the function F_P defined in Subsection 3.1, we have*

$$(4.3) \quad |F_{P_2}(P_1)| \leq \begin{cases} 3d^2 R^{12d^2+2} q_1^{-1} c, & \text{if } k = 1 \\ 3d^2 R^{k+d+1} q_1^{-1} c, & \text{if } k \geq 2 \end{cases}$$

Proof. Since $P_1, P_2 \in \mathcal{P}_{n+k,k}(B)$, we can pick points

$$\mathbf{x} = (x_1, \dots, x_d) \in \Delta_c(P_1) \cap B \quad \text{and} \quad \mathbf{x}' = (x'_1, \dots, x'_d) \in \Delta_c(P_2) \cap B.$$

Then,

$$\begin{aligned}
& |F_{P_2}(P_1)| \\
&= \left| \sum_{1 \leq i \leq d} a_{i,2} \frac{p_{i,1}}{q_1} - \sum_{1 \leq i \leq d} a_{i,2} \frac{p_{i,2}}{q_2} \right| \\
&= \left| \sum_{1 \leq i \leq d} a_{i,2} \left(\frac{p_{i,1}}{q_1} - x_i + x_i - x'_i + x'_i - \frac{p_{i,2}}{q_2} \right) \right| \\
&\leq \sum_{1 \leq i \leq d} |a_{i,2}| \left(\frac{c}{q_1^{1+r_i}} + \frac{c}{q_2^{1+r_i}} + 2R^{-n} \rho_0 \right) \\
&\leq \sum_{1 \leq i \leq d} q_2^{r_i} \left(\frac{c}{q_1^{1+r_i}} + \frac{c}{q_2^{1+r_i}} \right) + 2d\xi_{P_2} R^{-n} \rho_0 \quad (\text{by 3.9}) \\
&= \sum_{1 \leq i \leq d} \frac{cq_2^{r_i}}{q_1^{1+r_i}} + \frac{dc}{q_2} + \frac{2dR^{-n} \rho_0 H(P_2)}{q_2} \\
&\leq \frac{2dc}{q_1} \max\left\{ \frac{q_1}{q_2}, 1, \frac{q_2}{q_1} \right\} + \frac{2d^2 R^{k+1} c}{q_1} \cdot \frac{q_1}{q_2} \\
&\leq \begin{cases} (2d^2 R^{k+1} + 2d) R^{12d^2} q_1^{-1} c & \text{if } k = 1, \\ (2d^2 R^{k+1} + 2d) R^d q_1^{-1} c & \text{if } k \geq 2 \end{cases} \quad (\text{by definitions in Section 3.2}) \\
&\leq \begin{cases} 3d^2 R^{12d^2+2} q_1^{-1} c & \text{if } k = 1, \\ 3d^2 R^{k+d+1} q_1^{-1} c & \text{if } k \geq 2. \end{cases}
\end{aligned}$$

□

Proof of Proposition 4.4. The proof is divided according to two cases.

In the case of $k = 1$, by (3.13) and (4.3) we have

$$q_1 |F_{P_2}(P_1)| \leq 3d^2 R^{12d^2+2} c < 1.$$

Since $q_1 |F_{P_2}(P_1)| \in \mathbb{Z}$, then $F_{P_2}(P_1) = 0$. Hence, P_1 lies on \mathcal{H}_{P_2} .

Now we assume that $k \geq 2$. Note that we have attached a rational line \mathcal{L}_P passing through P to each rational vector P with $k_P \geq 2$. Write the corresponding vector \mathbf{v}_{P_j} ($j = 1, 2$) as

$$\mathbf{v}_{P_j} = (v_{1,j}, \dots, v_{d,j}).$$

As $\mathbf{v}_{P_j} \in \Lambda_{P_j} \setminus \{\mathbf{0}\}$, there are $b_j \in \mathbb{Z}$, $\mathbf{z}_j = (z_{1,j}, \dots, z_{d,j}) \in \mathbb{Z}^d$ such that

$$(4.4) \quad \mathbf{v}_{P_j} = b_j \frac{\mathbf{p}_1}{q_1} + \mathbf{z}_j, \text{ or equivalently } v_{i,j} = b_j \frac{p_{i,1}}{q_1} + z_{i,j} \text{ where } i = 1, \dots, d.$$

We argue by contradiction. Suppose that P_1 does not lie on \mathcal{H}_{P_2} , or equivalently,

$$(4.5) \quad F_{P_2}(P_1) \neq 0.$$

Then, either \mathcal{L}_{P_1} is parallel to \mathcal{H}_{P_2} , or \mathcal{L}_{P_1} intersects with \mathcal{H}_{P_2} at a point.

If $k \geq 2$ and \mathcal{L}_{P_1} is parallel to \mathcal{H}_{P_2} , then

$$(4.6) \quad \mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1} = 0.$$

We claim that

$$(4.7) \quad q_1 v_{i,1} F_{P_2}(P_1) \in \mathbb{Z} \quad \text{for each } 1 \leq i \leq d.$$

In view of (4.4) and the definition of F_{P_2} , to prove (4.7) it suffices to show

$$(4.8) \quad b_1 q_1^{-1} \mathbf{a}_{P_2} \cdot \mathbf{p}_1 \in \mathbb{Z}.$$

This follows easily from (4.6).

As $\mathbf{v}_{P_1} \neq \mathbf{0}$, it follows from (4.5) and (4.7) that

$$(4.9) \quad q_1 |F_{P_2}(P_1)| \left(\sum_{1 \leq i \leq d} |v_{i,1}| \right) \geq 1.$$

Note that, by definition we have

$$(4.10) \quad \max_{1 \leq i \leq d} \{w_{i,1} q_1^{-r_i}\} = \max\{q_1^{-s}, \psi_{P_1}\} \leq \psi_{P_1}.$$

Hence we have

$$\begin{aligned} & q_1 |F_{P_2}(P_1)| \left(\sum_{1 \leq i \leq d} |v_{i,1}| \right) \\ & \leq 3d^2 R^{k+d+1} c \left(\sum_{1 \leq i \leq d} (d-1) w_{i,1} q_1^{-r_i} \right) \quad (\text{by (3.20) and (4.3)}) \\ & \leq 3d^4 R^{k+d+1} c \max_{1 \leq i \leq d} \{w_{i,1} q_1^{-r_i}\} \\ & \leq 3d^4 R^{k+d+1} \psi_{P_1} c \quad (\text{by (4.10)}) \\ & \leq 3d^4 R^{k+d+1-dk-10d^2} c \quad (\text{by (3.18)}) \\ & < 1, \end{aligned}$$

which leads to a contraction.

If $k \geq 2$ and \mathcal{L}_{P_1} intersects with \mathcal{H}_{P_2} , let

$$P_0 = \frac{\mathbf{p}_0}{q_0} = \left(\frac{p_{1,0}}{q_0}, \dots, \frac{p_{d,0}}{q_0} \right)$$

be the point of their intersection. Write

$$(4.11) \quad \frac{\mathbf{p}_0}{q_0} = \frac{\mathbf{p}_1}{q_1} + \lambda_0 \mathbf{v}_{P_1}.$$

Now we are going to prove that

$$(4.12) \quad \Delta_c(P_1) \subsetneq \Delta_c(P_0),$$

by which we get a contradiction from the assumption that $P_1 \in \mathcal{C}_{n+k,k} \subset \mathcal{C}$. Applying the function F_{P_2} to both sides of (4.11), we get

$$(4.13) \quad \lambda_0 = - \frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}}.$$

Hence for $1 \leq i \leq d$, we have

$$(4.14) \quad \frac{p_{i,0}}{q_0} = \frac{p_{i,1}}{q_1} - \frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}} v_{i,1} = \frac{\sum_{1 \leq j \leq d} a_{j,2} (p_{i,1} v_{j,1} - p_{j,1} v_{i,1}) - q_1 v_{i,1} C_{P_2}}{q_1 \mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}}.$$

Now we claim that both the denominator and numerator of the fraction on the right hand side of (4.14) are integers.

Indeed, by (4.4), we have

$$p_{i,1} v_{j,1} - p_{j,1} v_{i,1} = p_{i,1} z_j - p_{j,1} z_i \in \mathbb{Z}.$$

Both the terms $q_1 v_{i,1}$ and $q_1 \mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}$ are easily seen to be integers by (4.4). This completes the proof of our claim. Then it follows that

$$(4.15) \quad q_0 \leq q_1 |\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}|.$$

We have

$$\begin{aligned}
& |\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}| \\
&= \left| \sum_{1 \leq i \leq d} a_{i,2} v_{i,1} \right| \\
&\leq \sum_{1 \leq i \leq d} \min\{q_2^{r_i}, \xi_{P_2}\} \cdot (d-1) w_{i,1} q_1^{-r_i} \quad (\text{by (3.20)}) \\
&= \sum_{1 \leq i \leq d} \min\{q_1^{-r_i} q_2^{r_i}, q_1^{-r_i} \xi_{P_2}\} \cdot (d-1) w_{i,1} \\
&\leq \sum_{1 \leq i \leq d} d \min\{R^{dr_i}, R^{dr_i} q_2^{-1-r_i} H(P_2)\} w_{i,1} \quad (\text{by definitions in Section 3.2}) \\
&\leq \sum_{1 \leq i \leq d} d R^d \min\{1, q_2^{-1-r_i} H(P_2)\} w_{i,1} \\
&\leq d^2 R^d \max\{\psi_{P_1}, \psi_{P_2}\} \quad (\text{by definitions of } \psi_P) \\
&\leq d^2 R^{d-dk-10d^2} \quad (\text{by (3.18)}) \\
(4.16) &< R^{-dk-6d^2}.
\end{aligned}$$

In particular

$$(4.17) \quad \frac{q_0}{q_1} \leq |\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}| < R^{-dk-6d^2} < R^{-1} < \frac{1}{2}.$$

For any $1 \leq i \leq d$, we have

$$\begin{aligned}
& q_0^{1+r_i} \left| \frac{p_{i,0}}{q_0} - \frac{p_{i,1}}{q_1} \right| \\
&= q_0^{1+r_i} \left| \frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}} v_{i,1} \right| \quad (\text{by (4.11) and (4.13)}) \\
&\leq q_1 |F_{P_2}(P_1)| \cdot |\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}|^{r_i} \cdot (d-1) w_{i,1} \quad (\text{by (3.20) and (4.15)}) \\
&\leq d q_1 |F_{P_2}(P_1)| \cdot \max\{|\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}|^s, \psi_{P_1}\} \quad (\text{by definitions of } \psi_P) \\
&\leq (3d^2 R^{k+d+1} c) \cdot (R^{-dk-6d^2})^{\frac{1}{d}} \quad (\text{by (4.3) and (4.16)}) \\
&\leq 3d^3 R^{1-5d} c \\
(4.18) &\leq \frac{c}{2} \quad (\text{by (3.12)}).
\end{aligned}$$

Now we are ready to prove claim (4.12). Indeed, for any $\mathbf{x} = (x_1, \dots, x_d) \in \Delta_c(P_1)$ and any $1 \leq i \leq d$,

$$\begin{aligned}
& |q_0^{1+r_i} (x_i - \frac{p_{i,0}}{q_0})| \\
&\leq |q_0^{1+r_i} (x_i - \frac{p_{i,1}}{q_1})| + |q_0^{1+r_i} (\frac{p_{i,1}}{q_1} - \frac{p_{i,0}}{q_0})| \\
&< (\frac{q_0}{q_1})^{1+r_i} c + \frac{c}{2} \quad (\text{by 4.18}) \\
&< c \quad (\text{by 4.17}).
\end{aligned}$$

□

Corollary 4.6. *Let $n \geq 1$, $B \in \mathcal{B}_n$ and $k \geq 1$. There exists a hyperplane $E_k(B) \subset \mathbb{R}^d$ such that for any $P \in \mathcal{C}_{n+k,k}$,*

$$\Delta_c(P) \cap B \subset E_k(B)^{(R^{-(n+k)} \rho_0)}.$$

Proof. Choose

$$\tilde{P} = \frac{\tilde{\mathbf{p}}}{\tilde{q}} = (\frac{\tilde{p}_1}{\tilde{q}}, \dots, \frac{\tilde{p}_d}{\tilde{q}}) \in \mathcal{C}_{n+k,k}(B)$$

such that

$$\tilde{q} = q(\tilde{P}) = \min\{q(P) : P \in \mathcal{C}_{n+k,k}(B)\}.$$

Let

$$\mathcal{H}_{\tilde{P}} = \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : F_{\tilde{P}}(\mathbf{x}) = \sum_{1 \leq i \leq d} \tilde{a}_i x_i + C_{\tilde{P}} = 0\}$$

be the associated hyperplane. We show that $\mathcal{H}_{\tilde{P}}$ is the hyperplane $E_k(B)$ we need, in other words

$$\Delta_c(P) \cap B \subset \mathcal{H}_{\tilde{P}}^{(R^{-(n+k)}\rho_0)}.$$

For any $P \in \mathcal{C}_{n+k,k}$, if $P \notin \mathcal{C}_{n+k,k}(B)$, this assertion is trivial. If

$$P = \frac{\mathbf{P}}{q} = \left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) \in \mathcal{C}_{n+k,k}(B),$$

then $P \in \mathcal{H}_{\tilde{P}}$ by Proposition 4.4. For any $\mathbf{x} = (x_1, \dots, x_d) \in \Delta_c(P) \cap B$, its distance to the hyperplane $\mathcal{H}_{\tilde{P}}$

$$\begin{aligned} & \frac{1}{\sqrt{\sum_{1 \leq i \leq d} \tilde{a}_i^2}} |F_{\tilde{P}}(\mathbf{x})| \\ & \leq \frac{1}{\xi_{\tilde{P}}} \left| \sum_{1 \leq i \leq d} \tilde{a}_i \left(x_i - \frac{p_i}{q}\right) \right| \\ & \leq \frac{1}{\xi_{\tilde{P}}} \sum_{1 \leq i \leq d} \frac{|c\tilde{a}_i|}{q^{1+r_i}} \\ & \leq \frac{1}{\xi_{\tilde{P}}} \sum_{1 \leq i \leq d} \frac{c\tilde{q}^{r_i}}{q^{1+r_i}} \\ & \leq \frac{dc}{\tilde{q}\xi_{\tilde{P}}} \\ & \leq \frac{dc}{H_{n+k}} \\ & \leq R^{-(n+k)}\rho_0. \end{aligned}$$

This finishes the proof. \square

5. PROOF OF LEMMA 2.4

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4. It suffices to show that $\mathbf{Bad}_c(\mathbf{r})$ is (B_0, β, γ) -HPW. Denote the closed ball chosen by Bob at the i -th round as B_i with radius ρ_i . By [4, Remark 2.4], we may assume that $\rho_i \rightarrow 0$. In view of Lemma 3.2(1) and Lemma 4.3, we have

$$(5.1) \quad \mathbf{Bad}_c(\mathbf{r}) = \mathbb{R}^d \setminus \bigcup_{P \in \mathbb{Q}^d} \Delta_c(P) = \mathbb{R}^d \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{P \in \mathcal{C}_{n+k,k}} \Delta_c(P).$$

Recall the definition of \mathcal{B}_n from (3.10), as those \mathcal{B}_n are mutually disjoint, for each $i \geq 1$ there exists at most one $n \geq 1$ with $B_i \in \mathcal{B}_n$. According to the definition of (β, γ) -hyperplane potential game, we have $\rho_{i+1} \geq \beta\rho_i$. Hence for each $n \geq 1$, there exists an $i \geq 1$ with $B_i \in \mathcal{B}_n$. Let $i(n)$ denote the smallest i with $B_i \in \mathcal{B}_n$. Then, the map $n \mapsto i(n)$ is an injective map from $\mathbb{Z}_{\geq 1}$ to $\mathbb{Z}_{\geq 1}$. Let Alice play according to the following strategy: each time after Bob chooses a closed ball B_i , if $i = i(n)$ for some $n \geq 1$, then Alice chooses the family of hyperplane neighborhoods

$$\{E_k(B_{i(n)})^{(R^{-(n+k)}\rho_0)} : k \in \mathbb{N}\}.$$

where $E_k(B_{i(n)})$ is the hyperplane given in Corollary 4.6. Otherwise Alice makes an arbitrary legal move. Since $B_{i(n)} \in \mathcal{B}_n$, $\rho_{i(n)} > \beta R^{-n} \rho_0$. Then, (3.11) implies that

$$\begin{aligned} & \sum_{k=1}^{\infty} (R^{-(n+k)} \rho_0)^\gamma \\ &= (R^{-n} \rho_0)^\gamma (R^\gamma - 1)^{-1} \\ &\leq \left(\frac{\rho_i}{\beta}\right)^\gamma \left(\frac{\beta^2}{2}\right)^\gamma \\ &< (\beta \rho_i)^\gamma. \end{aligned}$$

Hence (2.9) is satisfied, and Alice's move is legal. According to (5.1) and Corollary 4.6, we have

$$\begin{aligned} & \bigcap_{i=0}^{\infty} B_i \\ &= \bigcap_{i=0}^{\infty} B_i \cap (\mathbf{Bad}(\mathbf{r}) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{P \in \mathcal{C}_{n+k,k}} \Delta_c(P))) \\ &\subset \mathbf{Bad}(\mathbf{r}) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{P \in \mathcal{C}_{n+k,k}} \Delta_c(P) \cap B_{i(n)}) \\ &= \mathbf{Bad}(\mathbf{r}) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{P \in \mathcal{C}_{n+k,k}(B_{i(n)})} \Delta_c(P) \cap B_{i(n)}) \\ &\subset \mathbf{Bad}(\mathbf{r}) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_k(B_{i(n)})^{(R^{-(n+k)} \rho_0)}). \end{aligned}$$

Thus the unique point $\mathbf{x}_\infty \in \bigcap_{i=0}^{\infty} B_i$ lies in

$$\mathbf{Bad}(\mathbf{r}) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_k(B_{i(n)})^{(R^{-(n+k)} \rho_0)}).$$

Hence, Alice wins. \square

REFERENCES

- [1] J. An, *Badziahin-Pollington-Velani's theorem and Schmidt's game*, Bull. Lond. Math. Soc. (2013), no. 4, 721-733.
- [2] J. An, *Two-dimensional badly approximable vectors and Schmidt's game*, Duke Math. J. **165** (2016), no. 2, 267-284.
- [3] J. An, V. Beresnevich, S. Velani, *Badly approximable points on planar curves and winning*, Preprint, arXiv:1409.0064.
- [4] J. An, L. Guan, D. Kleinbock, *Bounded orbits of diagonalizable flows on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$* , Internat. Math. Res. Notices. (2015), no. 24, 13623-13652.
- [5] D. Badziahin, A. Pollington, S. Velani, *On a problem in simultaneous Diophantine approximation: Schmidt's conjecture*, Ann. of Math. (2) **174** (2011), no. 3, 1837-1883.
- [6] V. Beresnevich, *Badly approximable points on manifolds*, Invent. Math. **202** (2015), no. 3, 1199-1240.
- [7] R. Broderick, L. Fishman, D. Kleinbock, A. Reich, B. Weiss, *The set of badly approximable vectors is strongly C^1 incompressible*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 2, 319-339.
- [8] S.G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. **359** (1985), 55-89.
- [9] L. Fishman, D. Simmons, M. Urbański, *Diophantine approximation and the geometry of limit sets in Gromov hyperbolic metric spaces*, Preprint, arXiv:1301.5630.
- [10] D. Kleinbock, *Flows on homogeneous spaces and Diophantine properties of matrices*, Duke Math. J. **95** (1998), no. 1, 107-124.

- [11] D. Kleinbock, B. Weiss, *Values of binary quadratic forms at integer points and Schmidt games*, Recent Trends in Ergodic Theory and Dynamical Systems (Vadodara, 2012) (2013), 77-92.
- [12] C.T. McMullen, *Winning sets, quasiconformal maps and Diophantine approximation*, Geom. Funct. Anal. **20** (2010), no. 3, 726-740.
- [13] E. Nesharim, D. Simmons, *$\mathbf{Bad}(s, t)$ is hyperplane absolute winning*, Acta Arith. 164 (2014), no. 2, 145-152.
- [14] W.M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178-199.
- [15] W.M. Schmidt, *Badly approximable systems of linear forms*, J. Number Theory **1** (1969), 139-154.
- [16] W.M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics **785**, Springer, Berlin, 1980.
- [17] W.M. Schmidt, *Open problems in Diophantine approximation*, in "Diophantine approximations and transcendental numbers (Luminy, 1982)", Progr. Math. **31**, Birkhäuser, Boston, 1983, pp. 271-287.

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